Music and Mathematics

From Pythagoras to Fractals

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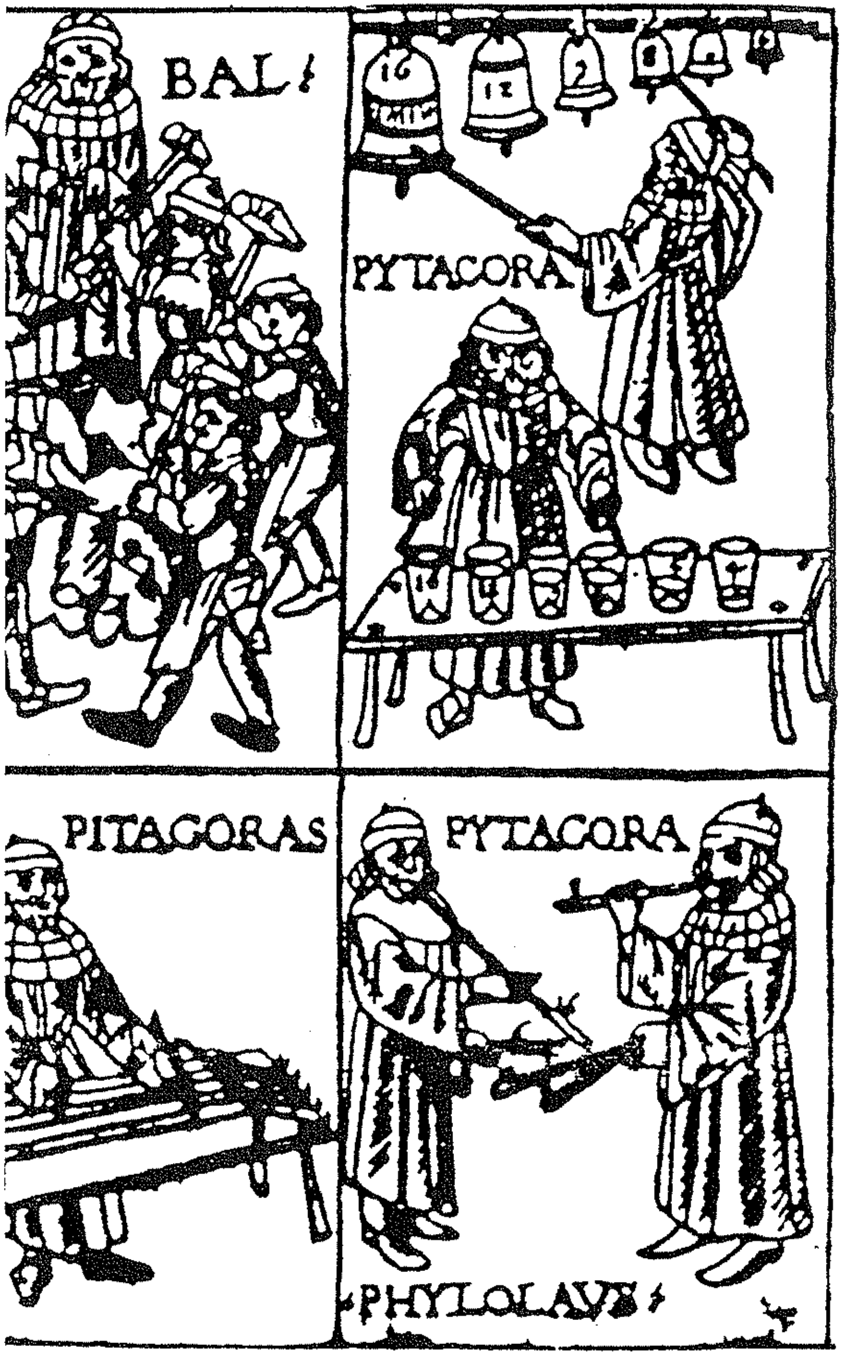
Preface

From ancient Greek times, music has been seen as a mathematical art. Some of the physical, theoretical, cosmological, physiological, acoustic, compositional, analytical and other implications of the relationship are explored in this book, which is suitable both for musical mathematicians and for musicians interested in mathematics, as well as for the general reader and listener.

In a collection of wide-ranging papers, with full use of illustrative material, leading scholars join in demonstrating and analysing the continued vitality and vigour of the traditions arising from the ancient beliefs that music and mathematics are fundamentally sister sciences. This particular relationship is one that has long been of deep fascination to many people, and yet there has been no book addressing these issues with the breadth and multi-focused approach offered here.

This volume is devoted to the memory of John Fauvel, Neil Bibby, Charles Taylor and Robert Sherlaw Johnson, whose untimely deaths occurred while this book was being completed.

Raymond Flood
February 2003 Robin Wilson



Ancient harmonic discoveries are portrayed in this woodcut from Franchino Gafurio's *Theorica musice* (1492). Mathematical ratios are emphasized in the experiments attributed to Pythagoras.

Tuning and temperament: closing the spiral

Neil Bibby

In Ancient Greek times it was recognized that consonant musical sounds relate to simple number ratios. Nevertheless, in using this insight to construct a scale of notes for tuning an instrument, problems arise. These problems are especially noticeable when transposing tunes so that they can be played in different keys. A solution adopted in European music over the last few centuries has been to draw upon mathematics in a different way, and to devise an 'equally-tempered' scale.

Each musical note has a basic frequency (essentially, the number of times the sound pulsates in a given period of time): thus the note 'A', which you may hear the oboe play while an orchestra is tuning up, has a frequency of 440 Hz (cycles per second). Frequency enables us to talk about relationships between musical sounds. However, for purposes of comparing two notes, the actual frequency is less important than the *ratio* of their frequencies.

The structure of a musical scale is determined by the frequency ratios of the notes that form the scale. The choice of these ratios is ultimately governed by the degree of *consonance* between the notes. Consonance is both a psychological and a physical criterion: two notes are *consonant* if they sound 'pleasing' when played together. In physical terms this seems to occur when the frequency ratio of the two notes is a ratio of low integers: the simpler the ratio, the more consonant are the two notes.

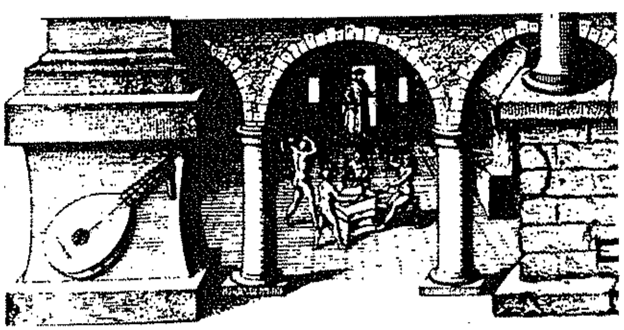
Apart from the trivial case of a unison, for which the frequency ratio is 1:1, the simplest case is the frequency ratio 2:1. When two notes have this frequency ratio the interval between them is an *octave*: thus, for the oboe A, the next higher A has frequency 880 Hz. The origins of this interval may lie in pre-history, when the earliest attempts at group singing or chanting would have been in unison, or in octaves for mixed groups of adults, or men and children: the different vocal ranges of the participants would thus force the harmonic use of the octave instead of the unison. As a melodic interval the octave is not common, but three popular twentieth-century American songs that start with a rising octave are *Somewhere over the rainbow, Singin' in the rain*, and *Bali Hai*.

This simple frequency relationship of 2 : 1, corresponding to two notes forming an octave, is the basis for the construction of any musical scale. Mathematically, the problem of constructing the scale is to determine an appropriate set of frequency ratios for the notes that lie in between. The number of these interpolated notes is arbitrary from a mathematical point of view. However, the frequency ratios of the intervening notes must satisfy the psychological/aesthetic criterion of consonance. Ultimately, as we shall see, the mathematical criterion of simplicity that underlies the early notion of consonance yields to other mathematical criteria. It turns out that the tolerance of the human ear, together with natural conditioning, enables the 'simplicity' criterion to be partially relaxed.

The Pythagorean scale

The oldest system of scale construction is that described as the *Pythagorean scale*. The system is much older than Pythagoras (*c*.550 BC), but his name is associated with the theoretical justification, in mathematical terms, of its construction. Legends have come down to us, through the late Roman writer Boethius among others, relating how Pythagoras 'discovered' this scale: they alleged that Pythagoras noted the harmonious relationships of the sounds produced by the hammers in a blacksmith's forge, and further investigations revealed that the masses of these hammers were, extraordinarily, in simple whole-number ratios to each other! From this claimed observation Pythagoras is supposed to have leapt to the realization that consonant sounds and simple number ratios are correlated—that ultimately music and mathematics share the same fundamental basis.

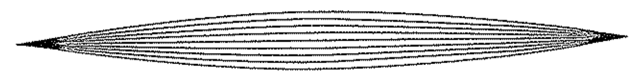
It is not difficult to construct a scale by following the Pythagorean insight. The strategy is to take any note and produce others related to



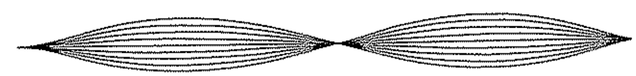
ert Fludd's *Temple of music*
), showing Pythagoras
cksmith's forge.

it by simple whole-number ratios, in the confidence that on Pythagorean principles the resultant notes will sound consonant. The structure of such a scale is ultimately based on the simple frequency ratios of 2 : 1 and 3 : 1.

In the case of a plucked or bowed string, different notes may be produced depending on how the string vibrates, and this too seems to follow the Pythagorean observation. Consider a vibrating string sounding a note of frequency t.



The same string can also vibrate at twice the original frequency, sounding the note of frequency $2t$. The *interval* between the new and original notes is given by the ratio of the frequencies, $2t : t$ or 2 : 1, an octave.



If the string were to vibrate with three times the original frequency, it would sound a note of frequency $3t$.

The interval between the notes of frequencies $3t$ and $2t$ is 3 : 2, or $\frac{3}{2}$. Equivalently, the note an octave below $3t$ is $\frac{3}{2}t$, and the interval between the note with frequency t and this note is therefore $\frac{3}{2}$.

We now have a three-note scale $\{t, \frac{3}{2}t, 2t\}$. If we regard the note with frequency t as the note C, for example, with C′ an octave higher, then this scale is

C	G	C′
t	$\frac{3}{2}t$	$2t$

This procedure has not only created a new note (G), but also a further new interval. Our previous interval, between C and G, is called a *perfect fifth* and the new interval between G and C′ is called a *perfect fourth*. The ratio corresponding to the perfect fifth is $\frac{3}{2}$, as we have seen, while the perfect fourth has ratio $2t : \frac{3}{2}t$, or $\frac{4}{3}$.

We now have a method for generating yet more notes. If we lower the note C′ by a perfect fifth, by dividing its frequency by $\frac{3}{2}$, we obtain the note F of frequency $\frac{4}{3}t$. It lies between C and G, and the resulting scale is

C	F	G	C′
t	$\frac{4}{3}t$	$\frac{3}{2}t$	$2t$

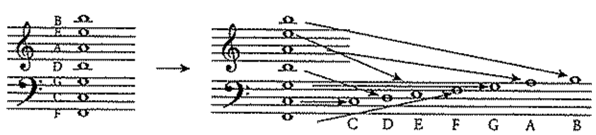
The process by which the scale is generated is thus essentially iterative: each new note yields a new interval with its nearest neighbour, and this interval can then be used to generate further new notes.

By continuing in this way, we obtain the interval between F and G. This is called the *major second*, or *whole tone*, and has ratio $\frac{3}{2}t : \frac{4}{3}t$, or $\frac{9}{8}$. This new interval in turn gives rise to a new note by simultaneously lowering both F and G by a perfect fourth: the new note, a whole tone above C, is D. We can now use the whole tone interval to fill in the gaps in the scale:

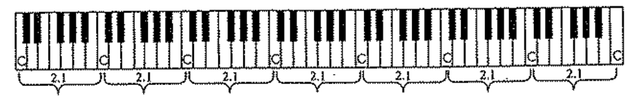
name of note	C	D	E	F	G	A	B	C′
frequency	$\frac{1}{1}t$	$\frac{9}{8}t$	$\frac{81}{64}t$	$\frac{4}{3}t$	$\frac{3}{2}t$	$\frac{27}{16}t$	$\frac{243}{128}t$	$\frac{2}{1}t$
interval		$\frac{9}{8}$	$\frac{9}{8}$	$\frac{256}{243}$	$\frac{9}{8}$	$\frac{9}{8}$	$\frac{9}{8}$	$\frac{256}{243}$

Each of the resulting 'narrow' intervals E to F and B to C is a *minor second*, or *semitone*, and has a ratio of $\frac{4}{3} : \frac{81}{64}$, which is $\frac{256}{243}$. In addition, several other new intervals appear, including the major third C to E, with ratio $\frac{81}{64}$, the major sixth C to A, with ratio $\frac{27}{16}$, and the major seventh C to B, with ratio $\frac{243}{128}$. We thus arrive at the Pythagorean scale, and we denote the resulting set of notes by **P**.

An alternative view is to regard the scale as being formed by a succession of perfect fifths, starting from C. In this view, we form the five notes that are successive fifths above C, and the note that is a perfect fifth below C. We then reassemble these into a single octave.



The result of this process is equivalent to our earlier one. In the resulting scale, successive notes are separated by an interval of a tone, with ratio $\frac{9}{8}$, or a semitone, with ratio $\frac{256}{243}$. The semitone is actually smaller than its name would suggest, because $(\frac{256}{243})^2$ is less than $\frac{9}{8}$—so it is not a 'semi'-tone in any accurate sense! We shall see later that this leads to serious problems: for example, on a modern keyboard it *seems* as though twelve perfect fifths are equivalent to seven octaves. However, if the tuning is Pythagorean, this cannot possibly be the case, as we shall see later.



More generally, if we stick to octaves and perfect fifths, then only the numbers 2 and 3 (and their powers) can be involved in these ratio calculations. Thus, each note in the Pythagorean scale can be written simply as $2^p \cdot 3^q$, where p and q are integers: here, and from now on, we omit the factor t. The scale **P** can thus be represented as follows:

C	D	E	F	G	A	B	C′
1	$3^2/2^3$	$3^4/2^6$	$2^2/3$	$3/2$	$3^3/2^4$	$3^5/2^7$	2

Exploring further the way that the notes of the Pythagorean scale combine, however, we run into a problem. Suppose that we wish to find the note a major seventh above A ($3^3/2^4$): this note is $3^3/2^4 \times 3^5/2^7 = 3^8/2^{11}$. Lowering this by an octave, we get $3^8/2^{12}$, which must lie somewhere between G and A (since $3/2 < 3^8/2^{12} < 3^3/2^4$). This leads us to realize that the Pythagorean scale is not 'closed' under transposition, but the rules under which we have constructed the scale will lead to an indefinite number of new notes. This leads to problems if we want to construct a scale (in particular, a physically embodied scale such as a keyboard) that allows transposition of keys.

Transposition in the Pythagorean scale

We constructed the Pythagorean scale **P** by a succession of transpositions of the basic key note C: in each case we transposed up a fifth (multiplying its frequency by $\frac{3}{2}$) and where necessary took the resulting note down an octave (halving its frequency). A good way of seeing what is going on in the problematic issue which has just arisen, of an apparently indefinite number of new notes being produced, is to consider the effect of the same transpositions on the entire scale **P**. Does this lead to another Pythagorean scale, and are the same notes involved?

Let us build a new scale on the note G. To do this, we transpose the original Pythagorean scale **P** up by a fifth, and transpose down an octave when necessary. The resulting scale $\mathbf{P}^1$ includes most of the notes of **P** itself, as a result of the partial regularity of the distribution of the intervals between the original notes:

[tone-tone-semitone]-tone-[tone-tone-semitone].

However, there is a 'new' element, the note $3^6/2^9$: this note lies between the two existing notes F and G, since $2^2/3 < 3^6/2^9 < 3/2$. This new note is the familiar F sharp, written F♯, and is required when we transpose from the scale of C to the scale of G. It does not lie symmetrically between F and G, however, since the interval $3^7/2^{11}$ between F and F♯ is slightly greater than the interval $2^8/3^5$ between F♯ and G.

	C		D		E	F	F♯	G		A		B	C′
P	1		$3^2/2^3$		$3^4/2^6$	$2^2/3$		$3/2$		$3^3/2^4$		$3^5/2^7$	2
P^1	1		$3^2/2^3$		$3^4/2^6$		$3^6/2^9$	$3/2$		$3^3/2^4$		$3^5/2^7$	2

In a similar way, a new scale can be built on the note F. In this case we *divide* the frequencies of each note by $\frac{3}{2}$, and where necessary transpose up an octave. This new scale, which we may call **P**$_1$, again contains a 'rogue' element, with frequency $2^4/3^2$, which is the familiar B flat, written B♭, of the key of F. Again, this new note lies between two existing notes, A and B, since $3^3/2^4 < 2^4/3^2 < 3^5/2^7$, and again not symmetrically since $2^8/3^5$ is less than $3^7/2^{11}$: thus, the new note is less than the geometric mean of the two notes each side of it.

	C		D		E	F		G		A	B♭	B	C′
P	1		$3^2/2^3$		$3^4/2^6$	$2^2/3$		$3/2$		$3^3/2^4$		$3^5/2^7$	2
P$_1$	1		$3^2/2^3$		$3^4/2^6$	$2^2/3$		$3/2$		$3^3/2^4$	$2^4/3^2$		2

Continuing in this way, we successively generate a new note between a pair of old notes, with each new note being slightly higher or lower than the geometric mean of its neighbours. After six such transpositions in each direction, we arrive at the scales **P**^{6} and **P**$_6$ opposite, in each row of which only one note (F or B, respectively) has survived from the original scale **P**.

The notes of the top row correspond to the key of F♯ and those of the bottom row correspond to that of G♭. By comparing these two scales, we can see that all of the notes of the G♭ scale are slightly lower than those of the F♯ scale. In particular, under the transposition into the key of F♯, the original key note C has become $3^6/2^9$, while under its transposition into G♭ it has become $2^{10}/3^6$. The interval between these notes is $(3^6/2^9)/(2^{10}/3^6)$, which simplifies to $3^{12}/2^{19}$ or 1.01364.... This very small difference, called the *Pythagorean comma*, lies at the root of the contradictions inherent in the Pythagorean scale. Although 3^{12} and 2^{19} are very close, they are not the same.

Furthermore, no succession of fifths can form an exact number of octaves—for if it did, there would be integer solutions p and q to the equation $(\frac{3}{2})^p = 2^q$, or $3^p = 2^{p+q}$. This has no solutions, since no power of 3 can equal a power of 2 (apart from the zeroth power), a particular

	C	C♯	D	D♯	E	F	F♯	G	G♯	A	A♯	B	C′	
P^6		$3^7/2^{11}$		$3^9/2^{14}$		$3^{11}/2^{17}$	$3^6/2^9$		$3^8/2^{12}$		$3^{10}/2^{15}$	$3^5/2^7$		F♯
P^5		$3^7/2^{11}$		$3^9/2^{14}$	$3^4/2^6$		$3^6/2^9$		$3^8/2^{12}$		$3^{10}/2^{15}$	$3^5/2^7$		B
P^4		$3^7/2^{11}$		$3^9/2^{14}$	$3^4/2^6$		$3^6/2^9$		$3^8/2^{12}$	$3^3/2^4$		$3^5/2^7$		E
P^3		$3^7/2^{11}$	$3^2/2^3$		$3^4/2^6$		$3^6/2^9$		$3^8/2^{12}$	$3^3/2^4$		$3^5/2^7$		A
P^2		$3^7/2^{11}$	$3^2/2^3$		$3^4/2^6$		$3^6/2^9$	$3/2$		$3^3/2^4$		$3^5/2^7$		D
P^1	1		$3^2/2^3$		$3^4/2^6$		$3^6/2^9$	$3/2$		$3^3/2^4$		$3^5/2^7$	2	G
P	1		$3^2/2^3$		$3^4/2^6$	$2^2/3$		$3/2$		$3^3/2^4$		$3^5/2^7$	2	C
P$_1$	1		$3^2/2^3$		$3^4/2^6$	$2^2/3$		$3/2$		$3^3/2^4$	$2^4/3^2$		2	F
P$_2$	1		$3^2/2^3$	$2^5/3^3$		$2^2/3$		$3/2$		$3^3/2^4$	$2^4/3^2$		2	B♭
P$_3$	1		$3^2/2^3$	$2^5/3^3$		$2^2/3$		$3/2$	$2^7/3^4$		$2^4/3^2$		2	E♭
P$_4$	1	$2^8/3^5$		$2^5/3^3$		$2^2/3$		$3/2$	$2^7/3^4$		$2^4/3^2$		2	A♭
P$_5$	1	$2^8/3^5$		$2^5/3^3$		$2^2/3$	$2^{10}/3^6$		$2^7/3^4$		$2^4/3^2$		2	D♭
P$_6$		$2^8/3^5$		$2^5/3^3$		$2^2/3$	$2^{10}/3^6$		$2^7/3^4$		$2^4/3^2$	$2^{12}/3^7$		G♭
	C	D♭	D	E♭	E	F	G♭	G	A♭	A	B♭	B	C′	

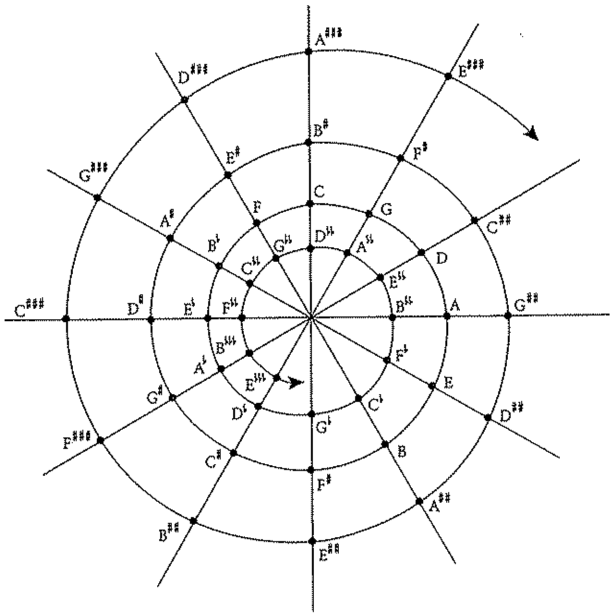
Pythagorean scales.

case of a mathematical result (the uniqueness of prime factorization) known since the time of Euclid. However, the fact that 3^{12} is approximately equal to 2^{19} suggests that $p = 12$, $q = 7$ is an approximate solution, and that the 'difference' can be measured by the ratio $3^{12}/2^{19}$, the Pythagorean comma.

We are thus faced with the fact that there is no end to the process we have initiated: transposition up a fifth and transposition down a fifth take us on infinite journeys, ever generating new notes, even if some of these (as with G♭ and F♯) are tantalisingly close. The journey can be thought of as traversing a spiral, starting from our set **P** (represented by C): for each 30° step clockwise we spiral outwards and transpose up a fifth, while for each 30° step anti-clockwise we spiral inwards and transpose down a fifth (see overleaf). Adjacent points on the same ray of the spiral differ by the Pythagorean comma.

Just intonation

Many of the intervals produced by the Pythagorean system are far from simple: what started as a system of consonances involving only small whole numbers has turned out to be less simple than at first appeared. For example the major third interval of $(\frac{9}{8})^2 = \frac{81}{64}$ and the major sixth $(\frac{27}{16})$ and the semitone $(\frac{256}{243})$ involve relatively large numbers. However, it



ıiral of Pythagorean fifths.

is important to note that musical intervals until the early renaissance were essentially melodic intervals: they would be perceived as relationships between *successive* notes, rather than as relationships between notes sounded *simultaneously*.

By the time of the early renaissance, polyphonic music had started to develop, and in addition to the harmonic use of octaves, fifths and fourths (hitherto, the only harmonic intervals generally employed), there was a gradual adoption of thirds and sixths. The use of these intervals involved a modification of the Pythagorean tuning under which the third ($\frac{81}{64}$) became slightly flattened to $\frac{80}{64}$, or $\frac{5}{4}$, and the sixth also became slightly flattened, from $\frac{27}{16}$ to $\frac{25}{15}$, or $\frac{5}{3}$.

During the sixteenth century, various attempts were made to modify the Pythagorean scale to incorporate these more consonant thirds and sixths. The most notable of the reformers was Giuseppe Zarlino, choirmaster at St Mark's in Venice. In 1558 he published *Institutioni harmoniche* in which he proposed an alternative mathematical basis for the major scale. He retained the Pythagorean relationships for the octave, fifth and tonic (4 : 3 : 2), but formalized the earlier *ad hoc* modification of the Pythagorean tuning by adopting the simpler relationships of 6 : 5 : 4 for the perfect fifth, major third and tonic—that is, $\frac{5}{4}$ for the major third and $\frac{6}{5}$ for the minor third. The scale he arrived at, known as the scale of

just intonation, was as follows:

note	C		D		E		F		G		A		B		C′
frequency	$\frac{1}{1}$		$\frac{9}{8}$		$\frac{5}{4}$		$\frac{4}{3}$		$\frac{3}{2}$		$\frac{5}{3}$		$\frac{15}{8}$		$\frac{2}{1}$
interval		$\frac{9}{8}$		$\frac{10}{9}$		$\frac{16}{15}$		$\frac{9}{8}$		$\frac{10}{9}$		$\frac{9}{8}$		$\frac{16}{15}$	

The frequencies of the notes of this scale can all be represented in the form $2^p \cdot 3^q \cdot 5^r$, where p, q and r are integers, and can be written as follows:

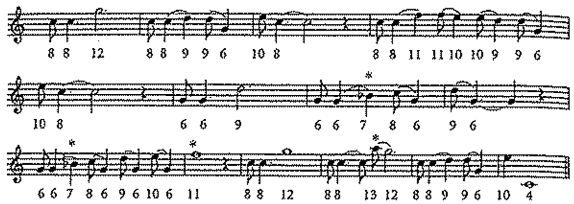
C	D	E	F	G	A	B	C′
1	$3^2/2^3$	$5/2^2$	$2^2/3$	$3/2$	$5/3$	$(3 \cdot 5)/2^3$	2

We shall refer to this set of notes as **J**. Several new intervals are produced by this scale. For instance, while there are Pythagorean whole tones ($\frac{9}{8}$) for C–D, F–G and A–B, ('major tones'), there are also narrower whole tones ('minor tones') for D–E and G–A of $\frac{10}{9}$. The ratio of these two intervals, $\frac{9}{8} : \frac{10}{9}$, the extent to which they are different tones, is called the *syntonic comma*: $\frac{81}{80} = 3^4/(2^4 \cdot 5) = 1.0125$, exactly.

The frequency ratios of the just intonation scale occur naturally in the 'harmonic series', and form the basis for playing certain wind instruments. Indeed, in the case of the horn, the technique of playing through using natural harmonics continued until valves were developed during the early nineteenth century. On the natural horn (without valves) the harmonics produce the following written notes.

In this sequence the 2nd, 4th and 8th harmonics correspond to the octave of the scale (that is, they are all the note C), and the 3rd, 6th and 12th harmonics sound G, the perfect fifth. The 9th harmonic sounds the major tone ($\frac{9}{8}$), which is the same in either Pythagorean or just intonation, whereas the 5th and 10th harmonics produce not the Pythagorean major third ($\frac{81}{64}$), but the just major third ($\frac{5}{4}$). Thus far, the natural harmonics are the same as just intonation. However, the 7th/14th, 11th and 13th harmonics (indicated with asterisks) produce notes of $\frac{7}{4}$, $\frac{11}{8}$ and $\frac{13}{8}$, which are wildly out of tune on either Pythagorean or just intonation. Players were expected to coax these notes into tune, the eleventh harmonic being flattened to F ($\frac{4}{3}$) and the seventh harmonic being sharpened up to B♭ ($\frac{16}{9}$). The English composer Benjamin Britten made extraordinary use of these notes in the solo horn prologue of his *Serenade for tenor, horn and strings*, which is scored for natural horn, or for an orchestral horn where the player does not use the valves; the harmonics are indicated in the figure overleaf.

Within a single scale, just intonation formed a reasonably satisfactory solution to problems thrown up by Pythagorean tuning, but the compromise breaks down when one wants to play in another key.

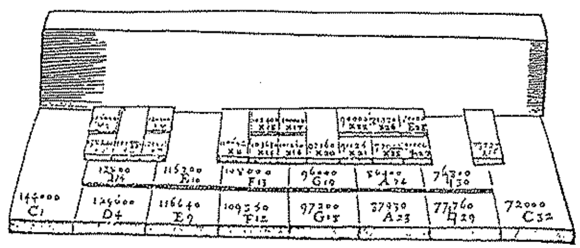


Prologue to Britten's *Serenade*.

Transposition with the just intonation scale is even more of a problem than for Pythagorean tuning. When we transpose up by a fifth, we find that the new scale includes *two* new notes: B is transposed to $F^{\sharp}$, as before, but the D also becomes a new note, an A of $3^3/2^4$, differing by a syntonic comma from the previous A of $\frac{5}{3}$. The reason for this is that the interval G-A in the original scale of C was a 'minor' tone, but became a 'major' tone after transposition.

	C		D		E	F	$F^{\sharp}$	G		A		B	C′
J	1		$3^2/2^3$		$5/2^2$	$2^2/3$		$3/2$		$5/3$		$3.5/2^3$	2
J¹	1		$3^2/2^3$		$5/2^2$		$3^2.5/2^5$	$3/2$		$3^3/2^4$		$3.5/2^3$	2

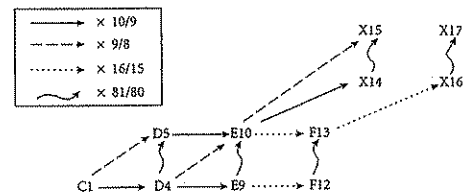
On fixed-pitch instruments, such as a harpsichord or organ, this situation made changes of key very difficult. Attempts to overcome the problem meant that alternative keys differing by a syntonic comma had to be provided. One seventeenth-century mathematician who took this issue seriously was Marin Mersenne. In the 31-note keyboard he described and discussed in his *Harmonie universelle* (1636–7), there were no fewer than four keys between F and G!



Mersenne's keyboard with 31 notes to the octave.

Two of these (X14 and X15) are G flats differing by a syntonic comma, one for each of the G naturals (again differing by a syntonic comma), one (X16) is an F sharp (for 'F13') and the fourth (X17) is a

syntonic comma higher than the F sharp of X16. The following diagram summarizes the relationship of these keys:

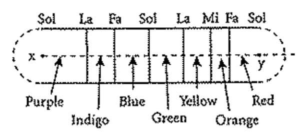


It is interesting to note that such keyboards were actually built: Handel, for example, played a 31-note organ in the Netherlands.

This multiplicity of keys is necessary because successive transpositions of the scale of just intonation generate even more notes between those of the basic set **J** than they did for the set **P**. In this case each transposition produces a new 'black' note, as in the Pythagorean case, but an extra new note is produced, a syntonic comma sharper for upward transpositions and flatter for downward. This arises, as we have seen, because one of the fifth intervals in the just scale is narrow—the interval D–A has ratio $\frac{5}{3} : \frac{9}{8}$ or $\frac{40}{27}$, which is less than $\frac{3}{2}$. In musical terminology, the old submediant is too flat to serve as the new supertonic.

The more transpositions take place, the worse the problems get. The effect of successive upward and downward transpositions of the basic just scale **J** is summarized overleaf.

In practice, modulations into remote keys were not usual at this time (partly, no doubt, for this reason): however, even to use the keys near to C in just intonation required two extra notes per modulation. The systems discussed so far imply infinitely many keys, with the spiral of fifths continuing infinitely, both outwards and inwards: the Pythagorean system $\mathbf{P}^*$, with notes generated by octaves and perfect fifths, and the just system $\mathbf{J}^*$, with notes generated by octaves, perfect fifths and major thirds, both yield infinite sets. So far as the construction of keyboard instruments was concerned, this was not an encouraging state of affairs.



Newton's spectrum scale.

Many attempts were made to develop tuning systems that overcame the difficulties of Zarlino's just system. Amongst these, Francesco Salinas (1530–90) proposed a system called *mean-tone*, in which the two whole tones of Zarlino's system ($\frac{9}{8}$ and $\frac{10}{9}$) were replaced by their geometric mean, thus giving a whole tone interval of $\frac{1}{2}\sqrt{5}$. The interval of the third remained a pure consonance of $\frac{5}{4}$, while the fifth had a ratio of $\sqrt[4]{5}$, which is approximately 1.4953: this is a little less than $\frac{3}{2}$, giving a rather flat fifth. Isaac Newton also spent much time trying to select the best ratios. Believing that seven notes in the octave and seven colours in the spectrum were too much of a coincidence, he even produced a

	C	C♯	D	D♯	E	F	F♯	G	G♯	A	A♯	B	C′	
J^6		$\frac{3^7}{2^{11}}$		$\frac{(3^5.5)}{2^{10}}$		$\frac{(3^7.5)}{2^{13}}$	$\frac{3^6}{2^9}$		$\frac{3^8}{2^{12}}$		$\frac{(3^6.5)}{2^{11}}$	$\frac{3^5}{2^7}$		F♯
J^5		$\frac{3^7}{2^{11}}$		$\frac{(3^5.5)}{2^{10}}$	$\frac{3^4}{2^6}$		$\frac{3^6}{2^9}$		$\frac{(3^4.5)}{2^8}$		$\frac{(3^6.5)}{2^{11}}$	$\frac{3^5}{2^7}$		B
J^4		$\frac{(3^3.5)}{2^{10}}$		$\frac{(3^5.5)}{2^{10}}$	$\frac{3^4}{2^6}$		$\frac{3^6}{2^9}$		$\frac{(3^4.5)}{2^8}$	$\frac{3^3}{2^4}$		$\frac{3^5}{2^7}$		E
J^3		$\frac{(3^3.5)}{2^7}$	$\frac{3^2}{2^3}$		$\frac{3^4}{2^6}$		$\frac{(3^2.5)}{2^5}$		$\frac{(3^4.5)}{2^8}$	$\frac{3^3}{2^4}$		$\frac{3^5}{2^7}$		A
J^2		$\frac{(3^3.5)}{2^7}$	$\frac{3^2}{2^3}$		$\frac{3^4}{2^6}$		$\frac{(3^2.5)}{2^5}$	$\frac{3}{2}$		$\frac{3^3}{2^4}$		$\frac{(3.5)}{2^3}$		D
J^1	1		$\frac{3^2}{2^3}$		$\frac{5}{2^2}$		$\frac{(3^2.5)}{2^5}$	$\frac{3}{2}$		$\frac{3^3}{2^4}$		$\frac{(3.5)}{2^3}$	2	G
J	1		$\frac{3^2}{2^3}$		$\frac{5}{2^2}$	$\frac{2^2}{3}$		$\frac{3}{2}$		$\frac{5}{3}$		$\frac{(3.5)}{2^3}$	2	C
J_1	1		$\frac{(2.5)}{3^2}$		$\frac{5}{2^2}$	$\frac{2^2}{3}$		$\frac{3}{2}$		$\frac{5}{3}$	$\frac{2^3}{3^2}$		2	F
J_2	1		$\frac{(2.5)}{3^2}$	$\frac{2^5}{3^3}$		$\frac{2^2}{3}$		$\frac{(2^3.5)}{3^3}$		$\frac{5}{3}$	$\frac{2^4}{3^2}$		2	B♭
J_3	$\frac{(2^4.5)}{3^4}$		$\frac{(2.5)}{3^2}$	$\frac{2^5}{3^3}$		$\frac{2^2}{3}$		$\frac{(2^3.5)}{3^3}$	$\frac{2^7}{3^4}$		$\frac{2^4}{3^2}$		$\frac{(2^5.5)}{3^4}$	E♭
J_4	$\frac{(2^4.5)}{3^4}$	$\frac{2^8}{3^5}$		$\frac{2^5}{3^3}$		$\frac{(2^6.5)}{3^3}$		$\frac{(2^3.5)}{3^3}$	$\frac{2^7}{3^4}$		$\frac{2^4}{3^2}$		$\frac{(2^5.5)}{3^4}$	A♭
J_5	$\frac{(2^4.5)}{3^4}$	$\frac{2^8}{3^5}$		$\frac{2^5}{3^3}$		$\frac{(2^6.5)}{3^5}$	$\frac{2^{10}}{3^6}$		$\frac{2^7}{3^4}$		$\frac{(2^8.5)}{3^6}$		$\frac{(2^5.5)}{3^4}$	D♭
J_6		$\frac{2^8}{3^5}$		$\frac{(2^9.5)}{3^7}$		$\frac{(2^6.5)}{3^5}$	$\frac{2^{10}}{3^6}$		$\frac{2^7}{3^4}$		$\frac{(2^8.5)}{3^6}$	$\frac{2^{12}}{3^7}$		G♭
	C	D♭	D	E♭	E	F	G♭	G	A♭	A	B♭	B	C′	

Just scales.

diagram linking the two; because he wished his scale to be symmetrical, he chose the note D as his starting point, obtaining the following scale:

note	D	E	F	G	A	B	C′	D′
frequency	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{2}{1}$
interval		$\frac{9}{8}$	$\frac{16}{15}$	$\frac{10}{9}$	$\frac{9}{8}$	$\frac{10}{9}$	$\frac{16}{15}$	$\frac{9}{8}$

Other compromise tunings were also developed, which incorporated some pure consonances: these sounded reasonably satisfactory for keys close to C, but in remote keys they could sound at best unsatisfactory, and at worst excruciating.

Equal temperament

By the beginning of the eighteenth century, it was beginning to be appreciated that for a keyboard to allow unlimited transposition, with no

key sounding more in tune than any of the others, it was necessary to divide the octave so that each note was generated by some basic interval: we call this a scale of *equal temperament*. Such ideas had been propounded long before this (in medieval China, for instance). More recently, Galileo Galilei's father Vincenso Galilei had proposed in *Dialogo della musica antica e moderna* (1581) that the scale be constructed from equal semitones with a frequency ratio of $\frac{18}{17}$. It is easy to check that $(\frac{18}{17})^{12}$ is about 1.9855 . . ., a little less than 2, and that $(\frac{18}{17})^7$ is about 1.4919 . . ., a little less than $\frac{3}{2}$. Such a scheme would therefore give rather flat octaves and flat fifths, hardly desirable features for the fundamental interval of any scale.

From this proposal it is but a short step to that of Simon Stevin (1548–1620), who suggested making the semitone interval equal to $2^{1/12}$, thereby preserving the octave's frequency ratio of 2. Since $2^{7/12} =$ 1.4983 . . ., this choice of semitone still gives slightly flat fifths, but better than those of Vincenso Galilei. $2^{1/12}$ is an irrational number, inexpressible as a fraction p/q and in addition, all of its powers up to the eleventh are also irrational. From a mathematical point of view this is ironical, given that we started out with a criterion for consonance essentially based on the notion of rationality. Of course, $2^{7/12}$ is an extraordinarily *good* approximation to $\frac{3}{2}$, so good that the difference is virtually imperceptible: herein lies the justification for its use. In the following table the frequency ratios for the major scale are compared in Pythagorean, just intonation and equal temperament:

	Pythagorean	*just intonation*	*equal temperament*
C	1	1	1
D	1.125	1.125	1.122462 . . .
E	1.265625	1.25	1.259921 . . .
F	1.333333 . . .	1.333333 . . .	1.334839 . . .
G	1.5	1.5	1.498307 . . .
A	1.6875	1.666666 . . .	1.681792 . . .
B	1.8984375	1.875	1.887748 . . .
C′	2	2	2

For ears accustomed to just intonation, the major third of almost 1.26 is noticeably sharp, and thus the extreme consonance of the just major chord (6 : 5 : 4) is lost in equal temperament.

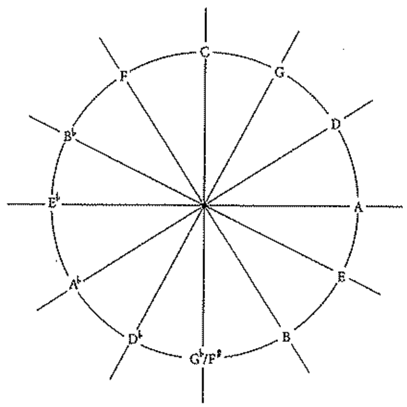
Under transposition, we can analyze the behaviour of the equal temperament scale in the same way as we did with the Pythagorean and just scales. The equally tempered major scale has the following notes:

C	D	E	F	G	A	B	C′
1	$2^{2/12}$	$2^{4/12}$	$2^{5/12}$	$2^{7/12}$	$2^{9/12}$	$2^{11/12}$	2

We can again apply the usual transpositions to this set; call it E, and let us trace what happens when we arrive at E^6 and E_6. In the following table $2^{1/12}$ is represented by α.

	C	C♯	D	D♯	E	F	F♯	G	G♯	A	A♯	B	C′
E^6		α^1		α^3		α^5	α^6		α^8		α^{10}	α^{11}	
E	1		α^2		α^4	α^5		α^7		α^9		α^{11}	2
E_6		α^1		α^3		α^5	α^6		α^8		α^{10}	α^{11}	
	C	D♭	D	E♭	E	F	G♭	G	A♭	A	B♭	B	C′

The 'new' notes (α^1, α^3, α^6, α^8, α^{10}, α^{11}) now sit symmetrically between the 'old' notes, since they are their geometric means. Hence the transposed sets are identical, so that the keys of G♭ and F♯ are the same. In this way the Pythagorean comma has now been eliminated, and the spiral has been closed into a circle. Six upward and six downward transpositions now give the same set of notes, and we thus arrive at the familiar 'circle of fifths':



Circle of fifths.

We call the set of notes thus obtained E*: the new notes obtained are those generated by $2^{1/12}$, because every note in E, E^6 or E_6 is some power of $2^{1/12}$. Moreover, any further transpositions can generate no new notes, so the set E* is a finite set. This is the great advantage of the equal temperament system: there are only twelve notes, and these allow unlimited transposition. The problem of keyboard design is thus solved, because each note now has infinitely many names: the key for F♯ is also that for G♭, as it is also for A𝄫 and E♯♯. By the removal of the Pythagorean comma, the spiral has indeed collapsed onto a circle.

The adoption of equal temperament was a lengthy process. Already in the late Elizabethan period (late sixteenth century) there is evidence that English virginal composers (notably John Bull) were modulating so far away from C that a form of equal temperament must have been in use, but as recently as the mid-nineteenth century it was by no means universal, especially in Britain: not one of the British organs at the Great Exhibition of 1851 was equally tempered. However, it is clear that during the early eighteenth century the system was increasingly being exploited. Fischer's *Ariadne musica* (1702), for instance, is a set of miniatures that go through nineteen of the twenty-four major or minor keys. The most famous work to exploit all twenty-four keys is J. S. Bach's *Well-tempered clavier* (1722 and 1738–44). Whether 'well-tempered' meant equally tempered in the modern sense is disputed, but the work includes a prelude and fugue for each major and minor key—hence the usual appellation of 'The 48 preludes and fugues'. Meanwhile a variety of compromise systems co-existed, including for instance the 'Kirnberger III system' which had four just tones, three mean tones, an equal-tempered fifth, nine different semitones and only four major seconds!

The fact that 2^{19} is nearly 3^{12}, and that $2^{7/12}$ is more-or-less $\frac{3}{2}$, is at the root of the equal temperament idea. The question naturally arises as to whether the approximate equation $2^p \approx 3^q$ has any other integer solutions, which might form the basis for an equally tempered scale that gives even better approximations to the just frequency ratios. There are infinitely many solutions, each corresponding to a rational approximation p/q of $\log_2 3$. A good example is $2^{84} \approx 3^{53}$, which leads to $2^{31/53} = 1.49994\ldots$, an excellent approximation to 1.5. This suggests that a structure of 53 notes to the octave (rather than 12) might be better for temperament purposes. In the nineteenth century R. Bosanquet actually made a harmonium with such a subdivision of the octave (see Chapter 5), and the twentieth century saw further exploration of this possibility. Of course, the development of electronic note production in the late twentieth century enabled completely accurate equally tempered systems with any number of notes, as we see in Chapter 9.

The idea of consonance is ultimately grounded in the notion of commensurability, an essential concept in Greek mathematics. We recognise consonance when we perceive a certain number of vibrations of one frequency exactly matching a certain number of another frequency. The Greeks accorded incommensurables a very different ontological status, and it thus remains a powerful irony that irrational numbers should come to the rescue—courtesy of the tolerance of the human ear and cultural conditioning—of the essentially rationally based system that they originally described for constructing a musical scale.